AFFINE REPRESENTATIONS OF THE FUNDAMENTAL GROUP (WITH AN APPENDIX ON PARABOLIC REPRESENTATIONS)

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ABSTRACT. Unitary representations of the fundamental group of a Kähler manifold correspond to polystable vector bundles (with vanishing Chern classes). Semisimple linear representations correspond to polystable Higgs bundles. In this paper we find the objects corresponding to affine representations: the linear part gives a Higgs bundle and the translation part corresponds to an element of a generalized de Rham cohomology.

1. INTRODUCTION

The affine group is the subgroup of $Gl_{n+1}(\mathbb{C})$

$$ Af_n(\mathbb{C}) = \left\{ \left(\begin{array}{cc} A & t \\ 0 & 1 \end{array} \right) : \quad A \in Gl_n(\mathbb{C}), \quad t \in Mat_{n \times 1} \right\}. $$

Given an element of $Af_n(\mathbb{C})$, the matrix $A \in Gl_n(\mathbb{C})$ is called the linear part and the column vector $t \in Mat_{n \times 1}$ is called the translation part. The map that gives the linear part defines a group homomorphism $Af_n(\mathbb{C}) \to Gl_n(\mathbb{C})$.

Recall that given a Kähler manifold X, the set of conjugacy classes of unitary representations of the fundamental group $\pi_1 \to U(n)$ is equal to the set of isomorphism classes of polystable rank n vector bundles with vanishing Chern classes ([NS], [D], [UY]). This correspondence can be extended to semisimple $Gl_n(\mathbb{C})$ representations. Then we have to consider polystable rank n Higgs bundles ([H], [S1], [S2], [S3]). In this paper we study affine representations.

Given an affine representation, the linear part gives a $Gl_n(\mathbb{C})$ representation, and if this is semisimple, this defines a polystable Higgs bundle $(\mathcal{E}, \theta)$. In this paper we prove that the extra object we have to add to the Higgs bundle to account for the translation part is an element of the first de Rham cohomology group $H^1_{DR}(E)$. This cohomology was introduced by Simpson in [S1].

From a different point of view, since $Af_n(\mathbb{C}) \subset Gl_{n+1}(\mathbb{C})$, an affine representation gives a $Gl_{n+1}(\mathbb{C})$ representation, but unless the translation part b is equal to zero, this representation won't be semisimple. Arbitrary (not necessarily semisimple) representations have been studied by Simpson in [S1] using differential graded categories. Roughly speaking, he shows that it is necessary to add a new field $\eta \in A^1(X, End(E))$, a smooth 1-form with values in $End(E)$, where E is the smooth vector bundle underlying the Higgs bundle corresponding to the induced semisimple representation.

From this point of view, what we show in this paper is that in the case of affine representations (assuming that the $Gl_n(\mathbb{C})$ representation given by the linear part A is semisimple) there is an explicit cohomological interpretation of η in terms of de Rham cohomology. If furthermore the linear part A is unitary, de Rham cohomology can be described in terms of the usual cohomology groups of coherent sheaves.

The parabolic construction of principal G-bundles on an elliptic curve done by [FMW] is related to this. Given a certain maximal parabolic subgroup P of G, with Levi factor L and unipotent part U, first they construct a principal L-bundle (the semisimple part of P), and then they show that the extra piece of data needed to specify the P-bundle is an element in the étale cohomology group $H^1_{et}(X, \underline{U})$, where $\underline{U}$ is the associated principal U-bundle. In the case $G = SL_n(\mathbb{C})$, this is a usual extension group Ext^1 of vector bundles.

The techniques used to study affine representations of the fundamental group can be adapted to consider representations into a parabolic subgroup of $Gl_n(\mathbb{C})$. The details are given in an appendix.

Notation. Holomorphic vector bundles will be denoted $\mathcal{E}$, $\mathcal{F}$, ... and the corresponding underlying smooth vector bundles will be denoted E, F, ... The trivial holomorphic line bundle is denoted $\mathcal{O}_X$, and the underlying smooth bundle O_X.

2. AFFINE BUNDLES

A holomorphic (resp. smooth) affine bundle is a holomorphic (resp. smooth) principal bundle with structure group $Af_n(\mathbb{C})$. Given a principal $Af_n(\mathbb{C})$-bundle, by the inclusion $Af_n(\mathbb{C}) \subset Gl_{n+1}(\mathbb{C})$ we obtain a principal $Gl_{n+1}(\mathbb{C})$-bundle. Since the transition functions $\{\alpha_{ij}\}$ of this principal bundle are of the form

$$(1) \qquad \alpha_{ij} = \begin{pmatrix} A_{ij} & t_{ij} \\ 0 & 1 \end{pmatrix},$$

the associated rank $n+1$ vector bundle $\mathcal{F}$ has a canonical rank n subbundle $\mathcal{E}$ (whose transition functions are $\{A_{ij}\}$), and the quotient is isomorphic to the trivial line bundle $\mathcal{O}_X$ (since the right lower entry of α_{ij} is equal to 1)

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_X \to 0.$$

Conversely, given an extension like this, we can find local trivializations of $\mathcal{F}$ such that the transition functions are of the form (1), and hence we obtain a principal $Af_n(\mathbb{C})$-bundle. It is easy to check that the isomorphism class of this affine bundle doesn't depend on the choice of local trivializations.

Definition 2.1. *Let $\underline{\mathcal{E}xtn}$ (resp. $\underline{Extn}$) be the category whose objects are holomorphic (resp. smooth) extensions of the form*

(2) $$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_X \to 0$$

(with $\mathrm{rk}(\mathcal{F}) = n+1$, $\mathrm{rk}(\mathcal{E}) = n$), and whose morphisms are pairs (φ, ψ) where $\varphi : \mathcal{E} \to \mathcal{E}'$ and $\psi : \mathcal{F} \to \mathcal{F}'$ are vector bundle morphisms and the following diagram commutes

(3)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \xrightarrow{i} & \mathcal{F} & \xrightarrow{p} & \mathcal{O}_X & \longrightarrow & 0 \\
& & \varphi \downarrow & & \psi \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{E}' & \xrightarrow{i'} & \mathcal{F}' & \xrightarrow{p'} & \mathcal{O}_X & \longrightarrow & 0.
\end{array}
$$

Lemma 2.2. *The category of holomorphic (resp. smooth) principal $Af_n(\mathbb{C})$-bundles is equivalent to the category $\underline{\mathcal{E}xtn}$ (resp. $\underline{Extn}$).*

$\square$

The proof is analogous to the proof of the fact that the category of principal $Gl_n(\mathbb{C})$-bundles is equivalent to the category of rank n vector bundles.

Remark 2.3. One could be tempted to relax condition (3), and to allow for an arbitrary isomorphism on $\mathcal{O}_X$, instead of requiring it to be the identity (as it is done for the definition of weak isomorphism of extensions). But this wouldn't give the right category. This is easily checked by looking at the case where the base space X is a point. The dimension of the automorphisms group should then be $n^2 + n = \dim(Af_n(\mathbb{C}))$. This is the dimension that we obtain with the definition that we have given, but if we allowed for an arbitrary isomorphism on $\mathcal{O}_X$, the dimension would be $n^2 + n + 1$.

But if we are only interested in the set of isomorphism classes, this distinction is not important, since any isomorphism of $\mathcal{O}_X$ is multiplication by scalar, and this can be absorbed by rescaling φ and ψ.

Remark 2.4. Abstractly, an affine space modeled on a vector space V is a set $\mathbb{A}$ together with a transitive and free action of V, and the affine group is the automorphism group of $\mathbb{A}$. Given an extension as in (2), note that $\mathcal{E} \cong p^{-1}(0) \subset \mathcal{F}$ is a bundle of vector spaces that acts (by addition) on $A = p^{-1}(1) \subset \mathcal{F}$

$$\mathcal{E} \times A \to A,$$

and the action commutes with projection to the base space X. Then we can think of A as a bundle of affine spaces. Conversely, given A and a vector bundle $\mathcal{E}$ acting on A, we can recover (up to isomorphism) the extension $\mathcal{F}$. This gives an equivalent description of an affine bundle.

Definition 2.5. *Let $\underline{Pairs}$ be the category whose objects are pairs $(\mathcal{E}, \chi)$ where $\mathcal{E}$ is a rank n vector bundle and $\chi \in H^1(\mathcal{E})$, and a morphism between $(\mathcal{E}, \chi)$ and $(\mathcal{E}', \chi')$ is a morphism of vector bundles $\varphi : \mathcal{E} \to \mathcal{E}'$ such that*

$$\begin{array}{ccc} H^1(\mathcal{E}) & \stackrel{H^1(\varphi)}{\longrightarrow} & H^1(\mathcal{E}') \\ \chi & \longmapsto & \chi' \end{array}.$$

An object (2) of $\underline{Extn}$ gives an element $\chi \in \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{E}) = H^1(\mathcal{E})$, and hence an element of $\underline{Pairs}$. Note that if $\lambda \neq 0$ then $(\mathcal{E}, \chi)$ and $(\mathcal{E}, \lambda\chi)$ are isomorphic (take $\varphi = \lambda \boldsymbol{I}_\mathcal{E}$). This category is not equivalent to the category of affine bundles, but we have

Proposition 2.6. *There is a natural bijection between the isomorphism classes of principal holomorphic $Af_n(\mathbb{C})$-bundles and the isomorphism classes of pairs $(\mathcal{E}, \chi)$, where $\mathcal{E}$ is a rank n vector bundle and $\chi \in H^1(\mathcal{E})$.*

Proof. Using lemma 2.2, the set of isomorphism classes of holomorphic principal $Af_n(\mathbb{C})$-bundles is equal to the set of isomorphism classes of the category $\underline{Extn}$. Two isomorphic extensions as in (3) give isomorphic pairs $(\mathcal{E}, \chi)$ and $(\mathcal{E}', \chi')$, because (3) implies that the image of χ in $H^1(\mathcal{E}')$ under the map induced by φ is χ'.

Conversely, two isomorphic pairs give two extensions (unique up to non-canonical isomorphism) that are isomorphic in the category $\underline{Extn}$. $\blacksquare$

Remark 2.7. This proposition is also valid in the smooth category. Note that in the smooth category short exact sequences are always split, although this splitting is not unique (in the smooth category $H^1(E)$ is zero since the sheaf of smooth sections of E is fine). This means that a smooth affine bundle has a (smooth) reduction of structure group to $Gl_n(\mathbb{C}) \subset Af_n(\mathbb{C})$, where this inclusion is given by

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix}$$

The existence of this reduction is equivalent to the fact that any affine bundle has a smooth section. From a topological point of view if we interpret an affine bundle, following remark 2.4, as a fibration of affine spaces over X the existence of the smooth section is a consequence of the contractibility of the fibers.

3. Flat affine bundles

A (smooth) flat affine bundle is a (smooth) principal $Af_n(\mathbb{C})$-bundle A with a connection D_A such that $D_A^2 = 0$. A morphism of flat affine bundles is a morphism $f : A \to A'$ of principal bundles such that the pullback of the connection on A' is equal to the connection on A.

Definition 3.1. *Let* $\underline{FlatExtn}$ *be the category whose objects are extensions*

(4) $$0 \to E \to F \to O_X \to 0$$

with $\mathrm{rk}(E) = n$ *and* $\mathrm{rk}(F) = n+1$, *together with a flat connection* D_F *on* F *that respects* E *(in the sense that the image of* $\mathcal{A}^0(X, E)$ *is in* $\mathcal{A}^1(X, E) \subset \mathcal{A}^1(X, F)$, *and then it induces a connections on* E *and* O_X*) and that induces the trivial connection* D_{O_X} *on* O_X. *A morphism in this category is a morphism in* $\underline{Extn}$ *with* $\varphi^* D_{F'} = D_F$.

An affine bundle gives an extension like (4) by lemma 2.2. A flat affine connection gives a connection

$$D_F : \mathcal{A}^0(X, F) \to \mathcal{A}^1(X, F)$$

that preserves E, and the induced connection on O_X is the trivial connection D_{O_X}, and then we get an object of $\underline{FlatExtn}$. In fact, this construction gives an equivalence of categories:

Lemma 3.2. *The category of flat affine bundles is equivalent to the category* $\underline{FlatExtn}$.

$\square$

If the induced flat connection D_E on E is semisimple, then by [S1, Theorem 1] there is a harmonic metric on E and a decomposition $D_E = \nabla_E + \alpha$ where ∇_E is a unitary connection and $\alpha \in A^1(X, End(E))$ is a 1-form. Let ∂_E and $\overline{\partial}_E$ be the (1,0) and (0,1) part of ∇_E. We get a holomorphic vector bundle $\mathcal{E} = (E, \overline{\partial}_E)$. Since the metric is harmonic the 1-form can be written as $\alpha = \theta + \theta^*$ where $\theta \in H^0(End(\mathcal{E}) \otimes \Omega^1)$ is a holomorphic (1,0) form with values in $End(\mathcal{E})$ such that $\theta_{\wedge}\theta = 0$, and θ^* is the conjugate (0,1) form. Then θ is a Higgs field, and the Higgs bundle $(\mathcal{E}, \theta)$ is polystable. Following Simpson, we decompose $D'_E = \partial_E + \theta^*$, $D''_E = \overline{\partial}_E + \theta$, and define the following cohomology groups:

- The de Rham cohomology $H^i_{DR}(E)$ of a Higgs bundle is the cohomology of the complex $(A^\bullet(X, E), D)$, where $A^p(X, E)$ is the space of global p-forms with coefficients in E.
- The Dolbeault cohomology $H^i_{Dol}(E)$ is the cohomology of the complex $(A^\bullet(X, E), D''_E)$.
- The group $\mathbb{H}^i(E \otimes \Omega^\bullet)$ is defined as the hypercohomology of the complex

(5) $$E \xrightarrow{\theta_\wedge} E \otimes \Omega \xrightarrow{\theta_\wedge} E \otimes \Omega^2 \xrightarrow{\theta_\wedge} \cdots$$

Simpson shows in [S1] that these three cohomologies are naturally isomorphic.

Definition 3.3. *Let $\underline{Higgs\mathcal{E}xtn}$ be the category whose objects are pairs*

$$(0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_X \to 0, \Theta)$$

where the extension is an object of $\underline{\mathcal{E}xtn}$ such that the Chern classes of $\mathcal{F}$ vanish, and $\Theta \in H^0(End(\mathcal{F}) \otimes \Omega_X)$ is a Higgs field, such that $(\mathcal{E}, \Theta)$ is semistable, $\mathcal{E}$ is Θ-invariant, and the Higgs field induced on $\mathcal{O}_X$ is zero. A morphism in this category is a morphism in $\underline{\mathcal{E}xtn}$ with $\psi^*\Theta' = \Theta$.

Definition 3.4. *Let $\underline{\mathcal{DRH}}$ bet the category whose objects are triples $(\mathcal{E}, \theta, \xi)$, where $(\mathcal{E}, \theta)$ is a polystable Higgs bundle with vanishing Chern classes and $\xi \in H^1_{DR}(E)$. A morphism is a morphism φ of Higgs bundles such that*

$$\begin{array}{ccc} H^1_{DR}(E) & \xrightarrow{H^1_{DR}(\varphi)} & H^1_{DR}(E') \\ \xi & \longmapsto & \xi' \end{array}.$$

Theorem 3.5. *There are natural bijections among the following sets*

1. *The set of $Af_n(\mathbb{C})$ representations of the fundamental group such that the linear part $Gl_n(\mathbb{C})$ is semisimple, modulo conjugation by elements in $Af_n(\mathbb{C})$.*
2. *The set of isomorphism classes of objects of $\underline{FlatExtn}$ such that the induced flat connection D_E is semisimple*

$$\{(0 \to E \to F \to O_X \to 0, \ D_F) : D_E \text{ semisimple}\}/ \cong .$$

3. *The set of isomorphism classes of objects of $\underline{Higgs\mathcal{E}xtn}$ such that the induced Higgs bundle $(\mathcal{E}, \theta)$ is polystable*

$$\{(0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_X \to 0, \ \Theta) : (\mathcal{E}, \theta) \text{ polystable}\}/ \cong .$$

4. *The set of isomorphism classes of $\underline{\mathcal{DRH}}$,*

$$\{(\mathcal{E}, \theta, \xi)\}/ \cong .$$

Proof. $(1 \leftrightarrow 2)$ This follows from holonomy and lemma 3.2.

$(2 \leftrightarrow 4)$ Take and object in $\underline{FlatExtn}$, i.e. an extension

$$0 \to E \to F \xrightarrow{p} O_X \to 0$$

with a flat connection D_F preserving E. Take a C^∞ splitting $E \oplus O_X \cong F$. This is given by a smooth morphism $\tau : O_X \to F$ (i.e., a smooth section of F) with $p \circ \tau = \boldsymbol{I}_{O_X}$. The fact that D_F preserves E and that it induces the trivial connection D_{O_X} on O_X means that, using this splitting, D_F can be written as

$$\begin{pmatrix} D_E & b_\tau \\ 0 & D_{O_X} \end{pmatrix}$$

where D_E is the connection induced on E and b_τ is a smooth 1-form with values in E. Note that b_τ depends on the splitting (i.e. the smooth section τ chosen) but D_E doesn't. Flatness of D_F translates into

$$D_E^2 = 0 \quad \text{and} \quad D_E b_\tau = 0.$$

Since by hypothesis D_E is semisimple, as we have already explained it has a harmonic metric and then there is a polystable Higgs bundle $(\mathcal{E}, \theta)$ (this is independent of the chosen section τ, since D_E was). If we take a different splitting, i.e. change the section τ to a new smooth section σ of F (again with $p \circ \sigma = \boldsymbol{I}_{O_X}$), the diffeomorphism between the old and the new splitting is given by a matrix of the form

$$\left(\begin{array}{cc} \boldsymbol{I}_E & \tau - \sigma \\ 0 & 1 \end{array} \right)$$

Note that the image of $\tau - \sigma$ is in E (because $p \circ (\tau - \sigma) = 0$), and then this matrix makes sense. In the new splitting the connection D_F has the form

$$\left(\begin{array}{cc} D_E & b_\tau + D_F(\sigma - \tau) \\ 0 & D_{O_X} \end{array} \right)$$

Then the 1-form associated to the new splitting is $b_\sigma = b_\tau + D_F(\sigma - \tau)$. In other words, we obtain an element of $H^1_{DR}(E)$ (independent of the splitting).

To show that this construction gives a bijection, we now construct an inverse. Take a triple $(\mathcal{E}, \theta, \xi)$ where $(\mathcal{E}, \theta)$ is a polystable Higgs bundle and $\xi \in H^1_{DR}(E)$. Since the Higgs bundle is polystable, there is a harmonic metric that gives a flat connection D_E.

Let $b \in A^1(X, E)$ be a representative of ξ and define on $F = E \oplus O_X$ a connection

$$D_F = \left(\begin{array}{cc} D_E & b \\ 0 & D_{O_X} \end{array} \right).$$

$(4 \leftrightarrow 3)$ Let $(\mathcal{E}, \theta, \xi)$ be an object of $\underline{\mathcal{DRH}}$. Let $\overline{b} \in H^1_{Dol}(E)$ be the element corresponding to $\xi \in H^1_{DR}(E)$ under the natural isomorphism. Let $b \in A^1(X, E)$ be a D''_E-closed smooth 1-form representing $\overline{b}$. Then

$$(6) \qquad \theta_\wedge b^{1,0} = 0, \quad \overline{\partial}_E b^{1,0} + \theta_\wedge b^{0,1} = 0, \quad \text{and} \quad \overline{\partial}_E b^{0,1} = 0.$$

Consider the vector bundle $F = E \oplus O_X$. Using this splitting, let

$$\overline{\partial}_F = \left(\begin{array}{cc} \overline{\partial}_E & b^{0,1} \\ 0 & \overline{\partial}_{O_X} \end{array} \right) : A^0(X, F) \to A^{0,1}(X, F).$$

Equations (6) imply that $\overline{\partial}_F^2 = 0$, and then this defines a structure of holomorphic vector bundle $\mathcal{F}$. Let

$$\Theta = \left(\begin{array}{cc} \theta & b^{1,0} \\ 0 & 0 \end{array} \right) \in Hom(\mathcal{F}, \mathcal{F} \otimes \Omega).$$

Then $\overline{\partial}_{Hom(\mathcal{F}, \mathcal{F} \otimes \Omega)} \Theta = 0$ and $\Theta_\wedge \Theta = 0$ again by (6), and then $(\mathcal{F}, \Theta)$ is a Higgs bundle. It is easy to check that it is semistable (but not polystable if $b^{1,0} \neq 0$).

Conversely, given an object of $\underline{\mathcal{HiggsExtn}}$, let $F = E \oplus O_X$ be the underlying smooth vector bundle of $\mathcal{F}$, and let $\overline{\partial}_F$ be the corresponding $\overline{\partial}$-operator.

It can be written as

$$\overline{\partial}_F = \begin{pmatrix} \overline{\partial}_E & b_1 \\ 0 & \overline{\partial}_{O_X} \end{pmatrix} : A^0(X,F) \to A^{0,1}(X,F).$$

with $b_1 \in A^{0,1}(X,E)$ a smooth (0,1)-form. The fact that $\mathcal{E}$ is Θ-invariant and that it induces the zero Higgs bundle on $\mathcal{O}_X$ imply that Θ can be written as

$$\Theta = \begin{pmatrix} \theta & b_2 \\ 0 & 0 \end{pmatrix} \in Hom(\mathcal{F}, \mathcal{F} \otimes \Omega),$$

with θ a Higgs field on $\mathcal{E}$, and $b_2 \in A^{1,0}(X,E)$ a smooth (1,0)-form. Let $b = b_1 + b_2$. Since $(\mathcal{F}, \Theta)$ is a Higgs bundle, we have $\overline{\partial}_F^2 = 0$, $\overline{\partial}_{Hom(\mathcal{F}, \mathcal{F} \otimes \Omega)} \Theta = 0$ and $\Theta_\wedge \Theta = 0$, and this implies that $D''_E b = 0$, where $D''_E = \overline{\partial}_E + \theta$. Then b defines a class $\overline{b} \in H^1_{Dol}(E)$, and under the natural isomorphism we obtain an element $\xi \in H^1_{DR}(E)$.   □

The bijection between 2 and 3 can also be obtained using [S2, lemma 3.5]. We will finish this section with some remarks, but first we need the following lemma

Lemma 3.6. *If $\theta = 0$, then there is a natural isomorphism*

$$H^i_{Dol}(E) \cong \bigoplus_{j=0}^{i} H^j(\mathcal{E} \otimes \Omega^{i-j})$$

Proof. Since $\theta = 0$, $D''_E = \overline{\partial}_E$ and then

$$H^i_{Dol}(E) = \frac{\ker(A^i \longrightarrow A^{i+1})}{\mathrm{im}(A^{i-1} \longrightarrow A^i)} =$$

$$\frac{\oplus \ker(A^{i-j,j} \longrightarrow A^{i-j,j+1})}{\oplus \mathrm{im}(A^{i-j,j-1} \longrightarrow A^{i-j,j})} \cong \bigoplus_{j=0}^{i} H^j(\mathcal{E} \otimes \Omega^{i-j})$$

□

If we consider affine representations in which the linear part is unitary, then $\theta = 0$ and by the previous lemma $H^1_{Dol}(E) = H^1(\mathcal{E}) \oplus H^0(\mathcal{E} \otimes \Omega)$. Then such a representation corresponds to a triple $(\mathcal{E}, \xi_1, \xi_0)$ where $\xi_1 \in H^1(\mathcal{E})$ and $\xi_0 \in H^0(\mathcal{E} \otimes \Omega)$. Note that ξ_1 is the class of the extension in item 3 of theorem 3.5.

Metrics on holomorphic extensions and the corresponding metric connections have been studied in [BG] and [DUW]. Those connections are different from our connection. Since they consider unitary connections, if the extension is not trivial, their connection never respects the subbundle E (i.e. the image of $\mathcal{A}^0(X,E)$ is not in $\mathcal{A}^1(X,E)$), as opposed to what happens with our connection.

Finally let's compare theorem 3.5 with Simpson's extensions of Higgs bundles to include arbitrary (not semisimple) representations [S1]. Take an affine representation, and assume that its linear part is semisimple. Recall that the inclusion $Af_n(\mathbb{C}) \subset Gl_{n+1}(\mathbb{C})$ gives a $Gl_{n+1}(\mathbb{C})$ representation. Its semisimple part is just the linear part plus a trivial one-dimensional factor, so it gives a polystable Higgs bundle $(\mathcal{E}, \theta) \oplus (\mathcal{O}_X, 0)$. To take into account the non-semisimple part (the translation part in our case), Simpson adds a smooth 1-form $\eta \in A^1(X, End(E \oplus O_X))$ with $D'\eta = 0$ and $D''\eta + \eta_\wedge \eta = 0$. In theorem 3.5 we construct explicitly this form and show that it lies in $A^1(X, E) \subset A^1(X, End(E \oplus O_X))$, the "upper-right part" of the endomorphisms, and that it is a cocycle, hence defines an element of the cohomology group.

One can adapt theorem 3.5 to consider parabolic representations of the fundamental group, i.e. representations $\rho : \pi_1 \to P$ into a parabolic subgroup P of $Gl_n(\mathbb{C})$. If $P = U \cdot L$ is a Levi decomposition of P, the unipotent part U plays the same role as the translation part for affine representations and the Levi factor L corresponds to the "linear part". Instead of the short exact sequences considered on item 2, one has to consider filtrations of flat vector bundles. If the connections induced on the quotients are semisimple, they give polystable Higgs bundles. To recover the parabolic representation from this data, on has to add a 1-form, analogous to the 1-form defining the de Rham cohomology in item 4 of the theorem (this 1-form corresponds to the unipotent part of the representation). Equivalently, one can define a Higgs field as in item 3, obtaining a semistable (but in general not polystable) Higgs bundle. Details are given in the appendix.

4. Appendix: Parabolic representations

Let P be a parabolic subgroup of $Gl_n(\mathbb{C})$. In this appendix we will consider representations of the fundamental group into P.

Recall that there is a one to one correspondence between parabolic subgroups of $Gl_n(\mathbb{C})$ and flags in $\mathbb{C}^n$ (the parabolic group associated to a flag is the subgroup of $Gl_n(\mathbb{C})$ that respects the flag). Parabolic representations appear naturally when we consider non-semisimple linear representations. Let $\rho : \pi_1 \to Gl_n(\mathbb{C})$ be such a non-semisimple representation. Then there is a parabolic subgroup P of $Gl_n(\mathbb{C})$ such that ρ factors

$$\rho : \pi_1 \to P \subset Gl_n(\mathbb{C}),$$

and such that if the parabolic group P corresponds to a flag

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_p = \mathbb{C}^n,$$

then the associated representation ρ^{ss} in

$$\frac{V_1}{V_0} \oplus \frac{V_2}{V_1} \oplus \cdots \oplus \frac{V_p}{V_{p-1}}$$

is semisimple. Let $P = U \cdot L$ be a Levi decomposition, where U is the maximal unipotent subgroup of P and L is the Levi factor. In matrix form,

U and L are respectively matrices of the form

$$
\begin{pmatrix}
\boldsymbol{I}_1 & * & \cdots & * \\
 & \boldsymbol{I}_2 & \cdots & * \\
 & & \ddots & \vdots \\
0 & & & \boldsymbol{I}_p
\end{pmatrix}
\qquad
\begin{pmatrix}
*_1 & & & 0 \\
 & *_2 & & \\
 & & \ddots & \\
0 & & & *_p
\end{pmatrix}
$$

where $*_i$ is a matrix of dimension $\dim(V_i/V_{i-1})$, and $\boldsymbol{I}_i$ is the identity matrix of the same dimension. There is a projection $P \to L$, and the semisimple representation ρ^{ss} corresponds to the composition $\pi \to P \to L$.

Since the representation ρ^{ss} is semisimple, it gives a polystable Higgs bundle $(\mathcal{E}_1, \theta_1) \oplus (\mathcal{E}_2, \theta_2) \oplus \cdots \oplus (\mathcal{E}_p, \theta_p)$. We want to find what extra object we have to give to take into account the "non-semisimple part" U.

4.1. Nonlinear de Rham cohomology.

Let (E_i, D_i), $i = 1, \ldots, p$ be p flat bundles. Let $\widetilde{\underline{U}}$ be the subsheaf of $End(\oplus E_i)$ such that its local sections respect the filtration

$$
E_1 \subset E_1 \oplus E_2 \subset \cdots \subset (E_1 \oplus \cdots \oplus E_p)
$$

and such that they induce the zero endomorphism on each factor E_i. In other words, $\widetilde{\underline{U}}$ is the sheaf of Lie algebras of the sheaf of unipotent groups associated with this filtration.

A flat filtration of vector bundles with quotients (E_i, D_i) will be the following data:

- A filtration of vector bundles

$$
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_p = F.
$$

- Isomorphisms $F_i/F_{i-1} \cong E_i$
- A flat connection D_F on F that respects the filtration and such that it induces on each quotient E_i (via the previous isomorphisms) the connection D_i.

If we choose smooth splittings of the filtration (i.e. a smooth isomorphism $E_1 \oplus \cdots \oplus E_p \cong F$), then we can decompose $D_F = D_L + \eta$ where $D_L = D_1 \oplus \cdots \oplus D_p$ and $\eta \in A^1(X, \widetilde{\underline{U}})$ is a 1-form with values in $\widetilde{\underline{U}}$. In matrix form:

$$
D_F =
\begin{pmatrix}
D_1 & & & 0 \\
 & D_2 & & \\
 & & \ddots & \\
0 & & & D_p
\end{pmatrix}
+
\begin{pmatrix}
0 & \eta_{1,2} & \cdots & \eta_{1,p} \\
 & 0 & \cdots & \eta_{2,p} \\
 & & \ddots & \vdots \\
0 & & & 0
\end{pmatrix}
$$

Flatness of D_F translates into $D_L\eta + \eta_\wedge\eta = 0$. We will think of this as a nonlinear closedness condition. If we choose a different splitting, the new splitting is isomorphic to the old one by an isomorphism that has the matrix

form

$$M_0 = \begin{pmatrix} \boldsymbol{I}_1 & M_{1,2} & \cdots & M_{1,p} \\ & \boldsymbol{I}_2 & \cdots & M_{2,p} \\ & & \ddots & \vdots \\ 0 & & & \boldsymbol{I}_p \end{pmatrix}$$

and the new 1-form is $\tilde{\eta} = M_0^{-1} D_L M_0 + M_0^{-1} \eta M_0$. Define the nonlinear de Rham cohomology set as the set of (nonlinear) closed 1-forms modulo the equivalence relation generated by the change of splitting

$$H_{DR}^1(E_\bullet, D_\bullet) = \frac{\{\eta \in A^1(X, \widetilde{\underline{U}}) : \quad D_L \eta + \eta_\wedge \eta = 0\}}{\eta \sim M_0^{-1} D_L M_0 + M_0^{-1} \eta M_0}$$

Then the flat filtration F gives a well defined element of this cohomology. Note that this is a pointed set (the distinguished point being the point corresponding to $\eta = 0$). If $p = 2$ then it is easy to see that this is the (linear) de Rham cohomology $H_{DR}^1(Hom(E_2, E_1))$ defined by Simpson [S1].

There are two different natural definitions of morphism (and then two different notions of isomorphisms) for flat filtrations. Let $(F_\bullet, D_F)$ and $(F_\bullet', D_{F'})$ be two flat filtrations. A *strong* morphism (resp. isomorphism) of flat filtrations is a morphism (resp. isomorphism) $\varphi : F \to F'$ such that $\varphi^* D_{F'} = D_F$, (hence φ respects the filtration) and it induces the identity map on the quotients

$$\varphi_i : \frac{F_i}{F_{i-1}} \cong E_i \xrightarrow{\boldsymbol{I}} E_i \cong \frac{F_i'}{F_{i-1}'}$$

If the induced maps on quotients φ_i are isomorphisms with $\varphi_i^* D_i' = D_i$ (but φ_i is not necessarily the identity), then we say that φ is a *weak* morphism (resp. isomorphism).

It is easy to check that if F is strongly isomorphic to F', then the corresponding points in the nonlinear de Rham cohomology $H_{DR}^1(E_\bullet, D_\bullet)$ are the same $[\eta] = [\eta']$. If they are only weakly isomorphic, then if M is the induced map on 1-forms we have $[\eta'] = [M^{-1} D_L M + M^{-1} \eta M]$, but since the induced maps on the quotients E_i are not necessarily identities, this class is in general not equal to $[\eta]$.

Remark 4.1. If instead of flat connections we use $\overline{\partial}$-operators, we obtain the nonlinear $\overline{\partial}$-cohomology set $H_{\overline{\partial}}^1(E_\bullet, \overline{\partial}_\bullet)$, and this set parametrizes holomorphic filtrations with fixed quotients $(E_i, \overline{\partial}_i) = \mathcal{E}_i$. If $p = 2$ then this filtrations are just extensions of $\mathcal{E}_2$ by $\mathcal{E}_1$, we have $\eta_\wedge \eta = 0$, and $H_{\overline{\partial}}^1(E_\bullet, \overline{\partial}_\bullet) = H^1(Hom(\mathcal{E}_2, \mathcal{E}_1))$, the usual cohomology group of coherent sheaves.

4.2. **Main theorem for parabolic representations.**

Let $(\mathcal{E}_i, \theta_i)$, $i = 1, ..., p$ be a collection of polystable Higgs bundles with vanishing Chern classes. A filtered Higgs bundle with quotients $(\mathcal{E}_i, \theta_i)$ is a

Higgs bundle $(\mathcal{F}, \Theta)$ together with a holomorphic filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_p = \mathcal{F}$$

(such that the Higgs field Θ respects this filtration), and isomorphisms from $(\mathcal{E}_i, \theta_i)$ to the induced Higgs bundles on the quotients $\mathcal{F}_i / \mathcal{F}_{i-1}$. A weak isomorphism of filtered Higgs bundles is an isomorphism of Higgs bundles, respecting the filtrations, and inducing isomorphisms in the quotients. If it induces the identity morphisms on the quotients, then it is called a strong isomorphism.

Theorem 4.2. *There are natural bijections among the following sets*

1. *The set of P representations such that the induced L representation is semisimple, modulo conjugation by elements of P.*
2. *The set of weak isomorphism classes of flat filtrations of vector bundles inducing semisimple flat connections D_i on the quotients $E_i = F_i / F_{i-1}$*

$$\{(F_\bullet, D_F) : D_i \quad semisimple\} / \cong_{weak} .$$

3. *The set of weak isomorphism classes of filtered Higgs bundles inducing polystable Higgs bundles on the quotients $\mathcal{E}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$*

$$\{(\mathcal{F}_\bullet, \Theta) : (\mathcal{E}_i, \theta_i) \text{ polystable with vanishing Chern classes}\} / \cong_{weak} .$$

4. *The set of isomorphism classes of objects*

$$\{(\mathcal{E}_i, \theta_i), \xi\} / \cong$$

   where $(\mathcal{E}_i, \theta_i)$ are polystable Higgs bundles, $\xi \in H^1_{DR}(E_\bullet, D_\bullet)$ (where (E_i, D_i) is the flat bundle associated to the Higgs bundle $(\mathcal{E}_i, \theta_i)$ via a harmonic metric), and two such objects are considered isomorphic if there are isomorphisms $\psi_i : \mathcal{E}_i \to \mathcal{E}'_i$ of Higgs bundles sending ξ to ξ'.

Proof. $(1 \leftrightarrow 2)$ This is given by holonomy.

$(2 \leftrightarrow 3)$ Follows from [S1, lemma 3.5] (see the remarks after Simpson's proof).

$(2 \leftrightarrow 4)$ Note that since the flat filtration induces semisimple flat connections on the quotients E_i, we get polystable Higgs bundles $(\mathcal{E}_i, \theta_i)$, and as we have already discussed, the flat connection D_F gives a well defined element of $H^1_{DR}(E_\bullet, D_\bullet)$. It is easy to check that a weakly isomorphic flat filtration gives isomorphic Higgs bundles and element ξ'.

Conversely, given Higgs bundles $(\mathcal{E}_i, \theta_i)$ and ξ, define the filtration $F_\bullet$

$$0 \subset E_1 \subset E_1 \oplus E_2 \subset \cdots \subset (E_1 \oplus \cdots \oplus E_p).$$

Take a representative $\eta \in A^1(X, \widetilde{\underline{U}})$ of ξ, define the connection

$$D_F = (D_1 \oplus \cdots \oplus D_p) + \eta,$$

and this defines a flat filtration.

$\square$

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